# Math 3D MT Solutions

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Aaron Chen

## 1) Solve $yy' + x = 0,\ y(3) = -4$.

*Solution.* There's a couple perspectives.

1) Separate the equation into $yy' = -x$, which in Leibniz notation is $y\frac{dy}{dx} = -x$ so then

$$ydy = -xdx \longrightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C.$$

Because $C$ is just a constant, we can just simplify the equation into

$$y^2 = -x^2 + D.$$

Solving for the constant, we have $(-4)^2 = -(3^2) + D$, so $D = 16 + 9 = 25$. Then,

$$y^2 = 25 - x^2 \longrightarrow y = \pm\sqrt{25 - x^2}.$$

Since the initial condition lies on $x = 3$ with $y = -4$, we will take the (-) branch! Also, we need the square root to be non-negative, $25 - x^2 \geq 0$ which tells is $x^2 \leq 25, |x| \leq 5$. Hence,

$$y = -\sqrt{25 - x^2}, \quad \text{Domain of Validity: } (-5, 5).$$

2) Substitute $v = y^2$: So then $\frac{v'}{2} + x = 0 \iff v' = -2x$. This is separable again,

$$v = -x^2 + C.$$

Plugging back in for $v = y^2$, we have the same result $y^2 = -x^2 + C$ (with just a renamed constant). The rest of the 1st solution follows from here (continue from where $y^2 = -x^2 + D$ above).

Grading Checks (needed to show work):
. Separating or using substitution to start the problem and get a general solution
. Solving for the constant
. Solving for $y$ explicitly and choosing the (-) branch
. Finding the domain of validity.

## 2) General solution to $y' + \tan(x)y = \cos(x)$.

*Solution.* This is an integrating factor problem, with $R(x) = e^{\int \tan x\ dx} = e^{\ln|\sec x|}$. You may have remembered/memorized that $\int \tan x\ dx = \ln|\sec x|$. But this is why:

$$\int \tan x\ dx = \int \frac{\sin x}{\cos x} dx, \quad \text{Use } u = \cos x,\ du = -\sin x dx,$$

$$= \int \frac{-du}{u} = -\ln|u|, \quad \text{Plug back } u = \cos x,$$

$$= -\ln|\cos x| = \ln|[\cos x]^{-1}| = \ln|1/\cos x|$$

$$= \ln|\sec x| + C$$

where we don't include the constant for the integrating factor (or rather, we set $C = 0$ because it doesn't matter). Thus,

$$R(x) = e^{\int \tan x \; dx} = e^{\ln|\sec x|} = |\sec x| = \frac{1}{|\cos x|}.$$

Using this as our integrating factor, we remember that first multiply $R(x)$ to both sides,

$$\frac{y'}{|\cos x|} + \frac{\sin x}{\cos x|\cos x|} y = \frac{\cos x}{|\cos x|}.$$

Here, we note that regardless of the branch of absolute value we take, the sign is always going to cancel! For example, if we used $|\cos x| \to -\cos x$, the minus sign just cancels everywhere. Thus, we can just omit the absolute values,

$$\frac{y'}{\cos x} + \frac{\sin x}{\cos^2 x} y = 1.$$

Recalling that the integrating factor induces a product rule,

$$\frac{d}{dx}\left[\frac{1}{\cos x} y\right] = 1 \implies \frac{y}{\cos x} = x + C.$$

Hence, our general solution is

$$y(x) = (x + C)\cos x.$$

Grading Checks:
. Using integrating factor (correctly)
. Finding the integral of tangent for the integrating factor
. Clarifying why the sign is always positive
. Correct general solution

## 3) General solution to $y' = \frac{y + xe^{-y/x}}{x}$.

*Solution.* We notice that we should use the substitution $v = y/x$ as a homogeneous equation because the right hand side looks like some $F(y/x)$. We then recall that

$$v = y/x \iff y = xv, \longrightarrow y' = xv' + v.$$

So our ODE becomes

$$xv' + v = v + e^{-v} \iff xv' = e^{-v}.$$

This is now separable, with Leibniz notation this reads as

$$e^v \frac{dv}{dx} = \frac{1}{x} \iff e^v dv = \frac{dx}{x}.$$

Integrating,

$$\int e^v dv = \int \frac{dx}{x} + C \longrightarrow e^v = \ln|x| + C.$$

Thus, $v = \ln(\ln|x| + C)$ where plugging in that $v = y/x$, we have that our general solution is

$$y(x) = x\ln(\ln|x| + C).$$

Grading Checks:
. Using the homogeneous substitution
. Correctly solving using the substitution (separation + integration)
. Plugging back in $v = y/x$ and getting the correct general solution

**4) Solve $y'' + 4y' + 10y = 0$ with initial conditions $y(0) = 3,\ \ y'(0) = -2$.**

*Solution.* Starting out with the auxiliary (characteristic) equation from assuming $y = e^{rx}$, we have

$$r^2 + 4r + 10 = 0.$$

If you factored this: You get $(r+2)^2 + 6 = 0$ so then $r = -2 \pm i\sqrt{6}$.
Use the quadratic formula:

$$r = \frac{-4 \pm \sqrt{16-40}}{2} = \frac{-4 \pm \sqrt{-24}}{2} = -2 \pm i\sqrt{6}.$$

Either way, we can now read the general solution:

$$y(x) = e^{-2x}[A_1\cos(\sqrt{6}x) + A_2\sin(\sqrt{6}x)].$$

We now solve for the constants with the initial conditions. First,

$$y(0) = 3: \quad 3 = e^0[A_1\cos(0) + A_2\sin(0)] \iff 3 = A_1.$$

To get $A_2$ we need to differentiate (and don't forget we can plug in for $A_1$ too):

$$y'(x) = -2e^{-2x}[3\cos(\sqrt{6}x) + A_2\sin(\sqrt{6}x)] + e^{-2x}[-3\sqrt{6}\sin(\sqrt{6}x) + \sqrt{6}A_2\cos(\sqrt{6}x)]$$

so now we can use our other initial condition,

$$y'(0) = -2: \quad -2 = -2[3] + [\sqrt{6}A_2] \iff 4 = \sqrt{6}A_2 \iff A_2 = \frac{4}{\sqrt{6}}.$$

Thus, our solution after plugging in these constants is

$$y(x) = e^{-2x}[3\cos(\sqrt{6}x) + \frac{4}{\sqrt{6}}\sin(\sqrt{6}x)].$$

Grading Checks:
. Getting and solving the auxiliary equation
. Correct general solution from the roots of the auxiliary equation
. Solving for both of the constants using both of the initial conditions